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Perfect fluid in a conformally flat space time

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Abstract. We obtain a new class of conformally flat solutions for a perfect fluid having equation of state $p = \alpha w$.

1. Introduction

In a pioneer work, Fock (1955) obtained conformally flat solutions of Einstein's equations for dust-like matter. Later Vashchuk (1969) carried out a systematic investigation of four-dimensional conformally flat spaces as gravitational fields satisfying Einstein's equations for the same type of distribution.

In the present paper, we will investigate the problem of determining the conformally flat solutions of Einstein's equations for a fluid satisfying the equations of state $p = \alpha w$, where p is the pressure, w is the rest-energy density and α is a constant.

2. Einstein's equations

Einstein's field equations for a perfect fluid with rest-energy density w, pressure p and four-velocity u_i can be written as

$$G_{ij} = R_{ij} - \tfrac{1}{2}g_{ij}R = -\kappa T_{ij} = -\kappa[(w+p)u_iu_j - pg_{ij}], \tag{1}$$

where

$$u_iu^i = 1.$$

We consider that the equation of state of the fluid is

$$\alpha w = p, \tag{2}$$

where α is a positive constant.

Assuming a space-time which is conformally flat,

$$g_{ij} = e^{2\sigma}\eta_{ij},$$

where η_{ij} is the flat metric having signature -2 and $\sigma = \sigma(x^i)$, the field equations (1)

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with condition (2) reduce to (Eisenhart 1966)

$$\sigma_{ij} + \tfrac{1}{2}\eta_{ij}\Delta_1\sigma = \tfrac{1}{2}\kappa w[\tfrac{1}{3}\eta_{ij}\,\mathrm{e}^{2\sigma} - (\alpha+1)u_i u_j], \tag{3}$$

where

$$\sigma_{ij} = \sigma_{,ij} - \sigma_{,i}\sigma_{,j}, \qquad \Delta_1\sigma = \eta^{ij}\sigma_{,i}\sigma_{,j}.$$

Equation (3) may be written as the following set of equations:

$$\sigma_{ij} = -\tfrac{1}{2}\kappa w(\alpha+1)u_i u_j, \qquad i \neq j, \tag{4}$$

$$\sigma_{ii} + \tfrac{1}{2}e_i\Delta_1\sigma = \tfrac{1}{2}\kappa w[\tfrac{1}{3}\,\mathrm{e}^{2\sigma}e_i - (\alpha+1)u_i^2], \qquad i = j, \tag{5}$$

where $e_i = \pm 1$, $(e_1 = e_2 = e_3 = -1,\ e_4 = 1)$.

To solve this overdetermined system of equations, (4) and (5), we note first that it is possible to obtain the rest-energy density in terms of the conformal factor, σ from the trace of the field equations (3),

$$w = [6/\kappa(1-3\alpha)]\,\mathrm{e}^{-2\sigma}(\eta^{ab}\sigma_{,ab} + \Delta_1\sigma). \tag{6}$$

Using equations (5) and (6), we can obtain the four-velocity in terms of the conformal factor σ,

$$u_i = \left[\frac{\mathrm{e}^{2\sigma}e_i}{3(\alpha+1)} - \frac{\mathrm{e}^{2\sigma}(1-3\alpha)}{3(\alpha+1)}\left(\frac{\sigma_{ii} + \frac{1}{2}e_i\Delta_1\sigma}{\eta^{ab}\sigma_{,ab} + \Delta_1\sigma}\right)\right]^{1/2}. \tag{7}$$

Now using equations (4), (6) and (7), we can write the differential equation that the conformal factor has to satisfy as

$$\left[\sigma_{ii} + e_i\Delta_1\sigma\left(\frac{1}{2} - \frac{1}{1-3\alpha}\right) - \frac{e_i}{1-3\alpha}\eta^{ef}\sigma_{,ef}\right] \times \left[\sigma_{jj} + e_j\Delta_1\sigma\left(\frac{1}{2} - \frac{1}{1-3\alpha}\right) - \frac{e_j}{1-3\alpha}\eta^{cd}\sigma_{,cd}\right] = \sigma_{ij}^2, \tag{8}$$

for $i \neq j$.

The problem of finding conformally flat solutions of Einstein's field equations has been reduced to investigating solutions for σ in equations (8). The conformal factor, to be physically reasonable, has to define the rest-energy density (6) as a positive function and the four-velocity (7) to satisfy $u_i u^i = 1$.

Supposing that the conformal factor is independent of one of the coordinates x^a, which implies

$$\sigma_{ia} = 0,$$

equations (8) can be reduced to

$$e_a\left[\Delta_1\sigma\left(\frac{1}{2} - \frac{1}{1-3\alpha}\right) - \frac{1}{1-3\alpha}\eta^{ef}\sigma_{,ef}\right]\left[\sigma_{jj} + e_j\Delta_1\sigma\left(\frac{1}{2} - \frac{1}{1-3\alpha}\right) - \frac{e_j}{1-3\alpha}\eta^{cd}\sigma_{,cd}\right] = 0, \tag{9}$$

for $j \neq a$. Equation (9) can be satisfied only if

$$\Delta_1\sigma\left(\frac{1}{2} - \frac{1}{1-3\alpha}\right) - \frac{1}{1-3\alpha}\eta^{ef}\sigma_{,ef} = 0, \tag{10}$$

because the case

$$\sigma_{jj} + e_j\Delta_1\sigma\left(\frac{1}{2} - \frac{1}{1-3\alpha}\right) - \frac{e_j}{1-3\alpha}\eta^{cd}\sigma_{,cd} = 0$$

gives a contradiction to the other equations of (8). Relation (10) then allows us to write the rest-energy density (6) as

$$w = (3/\kappa)\, e^{-2\sigma} \Delta_1 \sigma. \tag{11}$$

We observe then that as σ must be real, it must be time-dependent, as otherwise w will be negative.

3. Solutions

3.1. $\alpha \neq \frac{1}{3}$

Now we look for solutions of the system (8) assuming that $\alpha \neq \frac{1}{3}$ and that the functional dependence of the conformal factor is known. Since the functional dependence is not unique, we study two possible choices.

3.1.1. *First choice*

$$\sigma = \sigma(S), \qquad S = -x_1^2 - x_2^2 - x_3^2 + x_4^2. \tag{12}$$

Substituting (12) into (8) we obtain

$$A^2 + 4A(\sigma'' - \sigma'^2)(x_i^2 e_i + x_j^2 e_j) = 0, \tag{13}$$

where

$$A = 2\sigma' + 4S\sigma'^2\left(\frac{1}{2} - \frac{1}{1-3\alpha}\right) - \frac{1}{1-3\alpha}(8\sigma' + 4S\sigma'') \tag{14}$$

and the prime means differentiation with respect to S. Since A is a function only of the parameter S, equation (13) is only satisfied if $A = 0$, which may be written

$$\left(1 - \frac{4}{1-3\alpha}\right)\frac{1}{S} + \left(1 - \frac{2}{1-3\alpha}\right)\sigma' - \frac{2}{1-3\alpha}\frac{\sigma''}{\sigma'} = 0. \tag{15}$$

The solution of (15) is

$$e^{2\sigma} = \left(C_1 + \frac{C_2}{S^{(1+3\alpha)/2}}\right)^{4/(1+3\alpha)}. \tag{16}$$

In the particular case when $\alpha = 0$, (16) reduces to Vashchuk's result (15). We can obtain w and u_i for the metric (16) applying equations (6) and (7):

$$w = \frac{12}{\kappa} C_2 \left[C_1 + \frac{C_2}{S^{(1+3\alpha)/2}}\right]^{-4/(1+3\alpha)-1} S^{-(1+3\alpha)/2-1}\left[\left(\frac{C_1}{C_2} S^{(1+3\alpha)/2} + 1\right)^{-1} - 1\right], \tag{17}$$

$$u_i = \frac{x_i}{\sqrt{S}}\left(C_1 + \frac{C_2}{S^{(1+3\alpha)/2}}\right)^{2/(1+3\alpha)}, \tag{18}$$

where the constants of integration C_1 and C_2 are chosen such that w is a positive function and $u_i u^i = 1$.

3.1.2. *Second choice*

$$\sigma = \sigma(Z), \qquad Z = k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4, \tag{19}$$

where k_i are arbitrary constants.

Substituting (19) into (8) we obtain

$$B^2+B(\sigma''\sigma'^2)(k_i^2e_i+k_j^2e_j)=0, \tag{20}$$

where

$$B=k^2\left(\frac{1}{2}-\frac{1}{1-3\alpha}\right)\sigma'^2-k^2\frac{1}{1-3\alpha}\sigma'', \qquad k^2=\eta^{ij}k_ik_j$$

and the primes here mean differentiation with respect to Z. Since B is a function only of the parameter Z, equation (20) can only be satisfied if $B=0$, or

$$\left(\frac{1}{2}-\frac{1}{1-3\alpha}\right)\sigma'^2-\frac{1}{1-3\alpha}\sigma''=0,$$

which has the solution

$$e^{2\sigma}=(Z+C_3)^{4/(1+3\alpha)}, \tag{21}$$

where C_3 is a constant of integration. For $\alpha=0$ equation (21) reduces to Vashchuk's result (17). For solution (21) we can find w and u_i from (6) and (7):

$$w=\frac{12k^2}{\kappa}\frac{1}{(1+3\alpha)^2}(Z+C_3)^{-4/(1+3\alpha)-2}, \tag{22}$$

$$u_i=\frac{k_i}{k^2}(Z+C_3)^{2/(1+3\alpha)}. \tag{23}$$

3.2. $\alpha=\frac{1}{3}$

In the particular case when $\alpha=\frac{1}{3}$ we have a photonic fluid for which the trace of the energy–momentum tensor vanishes; hence the trace of the field equations (3) reduces to

$$\eta^{ij}(e^{\sigma})_{,ij}=0. \tag{24}$$

Two possible solutions of (24) are

$$e^{2\sigma}=(C_1+C_2/S)^2, \tag{25}$$

$$e^{2\sigma}=(Z+C_3)^2. \tag{26}$$

Introducing (25) into the field equations (3), we find

$$w=(12\,e^{-3\sigma}C_2/\kappa S^2)(C_2\,e^{-\sigma}/S-1), \tag{27}$$

$$u_i=x_i\,e^{\sigma}/S^{1/2}, \tag{28}$$

and introducing (26) into the field equations (3), we obtain

$$w=(3k^2/\kappa)\,e^{-4\sigma}, \tag{29}$$

$$u_i=k_i\,e^{\sigma}/(k^2)^{1/2}. \tag{30}$$

The constants C_1, C_2 and C_3 have to be chosen such that w is a positive function and $u_iu^i=1$.

We observe, then, that the conformally flat metrics given by (16) and (21) are valid also for $\alpha=\frac{1}{3}$.

3.3. $\alpha = 1$

One can obtain a more general class of solutions for the case $\alpha = 1$. As the metric demands that the motion of the fluid is irrotational, the four-velocity can be expressed in terms of a function $\phi(x^i)$ as

$$u_i = \phi_{,i}/(\mathrm{e}^{-2\sigma}\eta^{ab}\phi_{,a}\phi_{,b})^{1/2}, \tag{31}$$

and w and p can be expressed as

$$w = p = \tfrac{1}{2}\,\mathrm{e}^{-2\sigma}\eta^{ab}\phi_{,a}\phi_{,b}. \tag{32}$$

With conditions (31) and (32) it is possible to prove that the field equations (3) reduce to the field equations of a massless scalar field (Tabensky and Taub 1973):

$$\sigma_{ij} + \tfrac{1}{2}\eta_{ij}\Delta_1\sigma = \tfrac{1}{2}\kappa\,(-\phi_{,i}\phi_{,j} + \tfrac{1}{6}\eta_{ij}\eta^{ab}\phi_{,a}\phi_{,b}), \tag{33}$$

for which the trace is given by

$$\eta^{ij}\sigma_{ij} + 2\Delta_1\sigma = -\tfrac{1}{6}\kappa\eta^{ij}\phi_{,i}\phi_{,j}. \tag{34}$$

We study a special class of solutions by imposing a functional relationship of the form

$$\sigma = \sigma(\phi). \tag{35}$$

Substituting (35) into the trace (34) we obtain

$$\eta^{ij}\phi_{,i}\phi_{,j}(6\sigma'' - 3\sigma'^2 + \kappa) = 0, \tag{36}$$

where the primes denote differentiation with respect to ϕ. Since $\eta^{ij}\phi_{,i}\phi_{,j} \neq 0$, we obtain the differential equation

$$6\sigma'' - 3\sigma'^2 + \kappa = 0. \tag{37}$$

The general solution of (37) is

$$\sigma = -\ln\left\{\frac{C\exp[(2\kappa/3)^{1/2}(\phi - \xi)] + D}{C\exp[-(2\kappa/3)^{1/2}\xi] + D}\right\} + \frac{1}{2}\sqrt{\frac{2\kappa}{3}}\,\phi, \tag{38}$$

where ξ is a particular value of ϕ, $\eta = (2\sigma')_\xi$ given by the boundary conditions, and

$$C = (\kappa/6)^{1/2} - \tfrac{1}{2}\eta, \qquad D = (\kappa/6)^{1/2} + \tfrac{1}{2}\eta. \tag{39}$$

The constants which appear in (38) have been fixed such that when $\phi = 0$ we have $\sigma = 0$.

From equations (33) and (37) we find

$$\tfrac{2}{3}\kappa\phi_{,i}\phi_{,j} + 2\sigma'\phi_{,ij} + \eta_{ij}\eta^{ab}\sigma''\phi_{,a}\phi_{,b} = 0. \tag{40}$$

One particularly simple solution of equation (40) can be obtained from (38) when $C = 0$, which reduces to (Penney 1976)

$$\sigma = \tfrac{1}{2}(2\kappa/3)^{1/2}\phi. \tag{41}$$

By the substitution of (41) into (40) we obtain

$$\phi_{,ij} + (2\kappa/3)^{1/2}\phi_{,i}\phi_{,j} = 0, \tag{42}$$

which has the general solution

$$\phi = (3/2\kappa)^{1/2}\ln(Z + C_3). \tag{43}$$

Substituting (43) into (41) we find the conformally flat solution

$$e^{2\sigma} = Z + C_3, \tag{44}$$

which is the same as relation (21) with $\alpha = 1$ when we assumed the functional relation (19). In the particular case when $k_1 = k_2 = k_3 = 0$, $k_4 \neq 0$ and $C_3 = 0$, (44) becomes

$$e^{2\sigma} = k_4 x_4,$$

which is a Robertson–Walker space-time.

For $C \neq 0$ we define (Som and Santos 1978)

$$4\left\{\frac{C}{D}\exp\left[-\left(\frac{2\kappa}{3}\right)^{1/2}\xi\right] + \frac{D}{C}\exp\left[\left(\frac{2\kappa}{3}\right)^{1/2}\xi\right] + 2\right\}^{-1} = \pm a^2, \tag{45}$$

which allows us to write equation (38) as

$$\sigma = \pm\frac{1}{2}\left(\frac{2\kappa}{3}\right)^{1/2}\phi - \ln\left[\frac{1 \pm (1 \pm a^2)^{1/2}}{2}\exp\left(\pm\frac{2\kappa}{3}\phi\right) \mp \frac{a^2}{2[1 \pm (1 \pm a^2)^{1/2}]}\right]. \tag{46}$$

Substituting (46) into (40) we obtain the solution

$$\phi = -\left(\frac{3}{2\kappa}\right)^{1/2} \ln\left\{\frac{C}{D}\exp\left[-\left(\frac{2\kappa}{3}\right)^{1/2}\xi\right] + 1\right\} - \left(\frac{3}{2\kappa}\right)^{1/2} \ln\left(\frac{1}{1 \pm a^2 F} - \frac{1}{2}\right), \tag{47}$$

where

$$F = \alpha \eta^{ij}(x_i + \xi_i)(x_j + \xi_j),$$

and α and ξ_i are constants of integration.

By substituting (47) into (31) and (32) we can obtain w and u_i for a perfect fluid in the conformally flat space-time (46). We find

$$w = \frac{3}{4\kappa} e^{-2\sigma}\left[\frac{1}{(1 \pm a^2 F)^{-1} - \frac{1}{2}}\right]^2 \frac{a^4 \alpha F}{(1 \pm a^2 F)^4},$$

$$u_i = (\alpha/F)^{1/2}(x_i + \xi_i),$$

where we need the condition $\alpha F > 0$ for w to be a positive function and u_i to satisfy $u_i u^i = 1$.

Conclusion

We have found a new class of conformally flat solutions for the problem of a fluid with the equation of state $p = \alpha w$. One recovers the solutions given by Vashchuk from these solutions simply by putting $\alpha = 0$. When $\alpha = 1$ our solution is equivalent to that given by Taub. Further, one can investigate the case of the relativistic fluid by putting $\alpha = \frac{1}{9}$ in metrics (16) and (21). The solution corresponding to disordered radiation ($\alpha = \frac{1}{3}$) is also presented.

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